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Upper bounds on T_c for many-body, ferromagnetic, Ising systems

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Abstract. A method used previously to obtain upper bounds on the critical temperature T_c for ferromagnetic Ising systems with pair interactions is generalised to treat systems with odd or even many-body interactions. The results are compared to recent work of Horiguchi and Morita who, using a different method, found bounds on T_c for systems with even many-body interactions. An interesting feature of the method presented here involves the new use of some correlation inequalities.

1. Introduction

Very recently Horiguchi and Morita (1985b) presented a method which establishes upper bounds on the critical temperature (hereafter T_c) of ferromagnetic Ising spin systems with even-spin interactions. We have recently presented (Monroe 1985) a method which we used to establish similar bounds on T_c for ferromagnetic Ising spin systems with only two-body interactions with the emphasis on situations with more than just nearest-neighbour interactions present. Here we extend this method to ferromagnetic Ising spin systems with many-body interactions where by many-body we mean three or more spins. Our method can be applied to systems with either even-body or odd-body interactions present. Throughout this paper T_c will be defined as the temperature above which the spontaneous magnetisation is zero.

In particular we first look at a system on a square lattice with nearest-neighbour pair interactions and four-body interactions on each elementary square of the lattice. This system was considered by Horiguchi and Morita (1985a, b) and we compare our bounds to theirs. As an example of a system with a many-body interaction having an odd number of spins interacting we look at a triangular lattice system with a three-body interaction on each elementary triangle of the lattice. This system was solved exactly by Baxter and Wu (1973).

The method we use here varies in only one essential aspect from our earlier analysis of pair interaction systems. This aspect involves a type of correlation inequality similar to that used by Messager and Miracle-Sole (1977). We emphasise the new features necessary for analysis of the many-body interaction systems and for more details concerning the general aspects we refer the reader to Monroe (1985). In § 2 we introduce the necessary notation and establish our bounds for the two systems mentioned above. In the proof of these T_c bounds we assume certain correlation inequalities which we then prove in § 3. Section 4 consists of a short conclusion.

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2. T_c bounds

We begin by considering the square lattice with periodic boundary conditions, a spin variable σ on each site with $\sigma = \pm 1$, and Hamiltonian

$$H = -J_2 \sum \sigma_i \sigma_j - J_4 \sum \sigma_i \sigma_j \sigma_k \sigma_l - h \sum \sigma_i \tag{2.1}$$

where the first sum is over all nearest-neighbour pairs, the second sum is over all four-body interactions where the four spins involved form an elementary square of the lattice and the third sum is over all sites of the system. J_2, J_4 and h are all greater than or equal to zero. We pick a site as the zeroth site and label the sites around it as shown in figure 1. Later we will need a concise notation to represent specific four-body interactions. The Greek symbols in figure 1 represent these, for example α represents the interaction involving σ_0, σ_1, σ_2 and σ_3.

The thermal average of some function of the σ, $f(\sigma)$, is

$$\langle f(\sigma)\rangle = \sum f(\sigma) \exp(-\beta H)\left(\sum \exp(-\beta H)\right)^{-1} \tag{2.2}$$

where the summations in both numerator and denominator are over all configurations of the system and where $\beta = 1/kt$. The method for obtaining the T_c bound requires that we set various interaction strengths equal to zero. We denote any such deleted interactions by a subscript on the brackets representing the thermal average, for example, $\langle f(\sigma)\rangle_{01,03,\alpha}$ is the thermal average of $f(\sigma)$ for a system where the pair interactions between sites 0 and 1, and 0 and 3, as well as the four-body interaction α have been deleted.

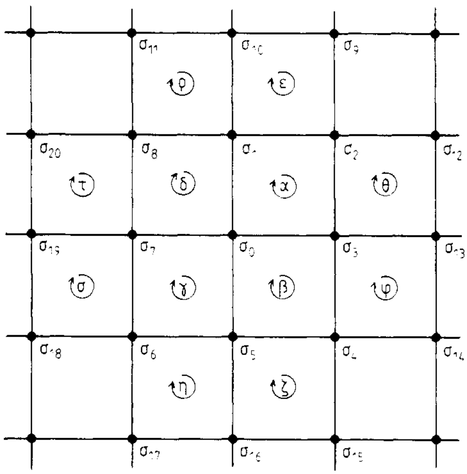

Figure 1. The square lattice with sites and four-body interactions labelled.

Following our earlier paper we can delete the pair interactions involving the zeroth site to obtain

$$\langle \sigma_0\rangle = \frac{\langle\sigma_0\rangle_{07,05,03,01}}{B_7B_5B_3B_1} + \frac{T_2\langle\sigma_7\rangle_{07,05,03,01}}{B_7B_5B_3B_1} + \frac{T_2\langle\sigma_5\rangle_{05,03,01}}{B_5B_3B_1} + \frac{T_2\langle\sigma_3\rangle_{03,01}}{B_3B_1} + \frac{T_2\langle\sigma_1\rangle_{01}}{B_1} \tag{2.3}$$

where

$B_7 = 1 + T_2\langle\sigma_0\sigma_7\rangle_{07,05,03,01}$ $\quad$ $B_5 = 1 + T_2\langle\sigma_0\sigma_5\rangle_{05,03,01}$ $\quad$ $B_3 = 1 + T_2\langle\sigma_0\sigma_3\rangle_{03,01}$

$B_1 = 1 + T_2\langle\sigma_0\sigma_1\rangle_{01}$ $\quad$ $T_2 = \tanh(\beta J_2)$.

Let A denote the last four terms of (2.3). Now we take $\langle\sigma_0\rangle_{07,05,03,01}$ and delete the four-body interaction terms involving the zeroth site to obtain

$$\langle\sigma_0\rangle=\frac{\langle\sigma_0\rangle'_{\delta,\gamma,\beta,\alpha}}{B_\delta B_\gamma\ldots B_3B_1}+\frac{T_4\langle\sigma_7\sigma_8\sigma_1\rangle'_{\delta,\gamma,\beta,\alpha}}{B_\delta B_\gamma\ldots B_3B_1}+\frac{T_4\langle\sigma_5\sigma_6\sigma_7\rangle'_{\gamma,\beta,\alpha}}{B_\gamma B_\beta\ldots B_3B_1}$$
$$+\frac{T_4\langle\sigma_3\sigma_4\sigma_5\rangle'_{\beta,\alpha}}{B_\beta B_\alpha\ldots B_3B_1}+\frac{T_4\langle\sigma_1\sigma_2\sigma_3\rangle'_\alpha}{B_\alpha B_7\ldots B_1}+A \tag{2.4}$$

where

$$B_\delta=1+T_4\langle\sigma_0\sigma_7\sigma_8\sigma_1\rangle'_{\delta,\gamma,\beta,\alpha} \qquad B_\gamma=1+T_4\langle\sigma_0\sigma_5\sigma_6\sigma_7\rangle'_{\gamma,\beta,\alpha}$$
$$B_\beta=1+T_4\langle\sigma_0\sigma_3\sigma_4\sigma_5\rangle'_{\beta,\alpha} \qquad B_\alpha=1+T_4\langle\sigma_0\sigma_1\sigma_2\sigma_3\rangle'_\alpha$$
$$T_4=\tanh(\beta J_4) \qquad \tau=\tanh(\beta h)=\langle\sigma_0\rangle'_{\delta,\gamma,\beta,\alpha}$$

and where to reduce the number of subscripts on the thermal average brackets we have used a prime to indicate deletion of the four pair interactions involving the zeroth site.

Now equation (2.4) can easily be turned into an upper bound on $\langle\sigma_0\rangle$ by setting each of the B, B_δ–B_1, equal to one. That each $B\geqslant 1$ is a direct consequence of the first Griffiths-Kelly-Sherman inequality (hereafter referred to as GKS I); see Griffiths (1967a) and Kelly and Sherman (1968). By the second Griffiths-Kelly-Sherman inequality (hereafter referred to as GKS II), see Griffiths (1967b) and Kelly and Sherman (1968), we know that adding ferromagnetic interactions to a system does not decrease the thermal average of any product of σ. Therefore we have

$$\langle\sigma_0\rangle\leqslant\tau+T_2\langle\sigma_7\rangle+T_2\langle\sigma_5\rangle+T_2\langle\sigma_3\rangle+T_2\langle\sigma_1\rangle+T_4\langle\sigma_7\sigma_8\sigma_1\rangle+T_4\langle\sigma_5\sigma_6\sigma_7\rangle$$
$$+T_4\langle\sigma_3\sigma_4\sigma_5\rangle+T_4\langle\sigma_1\sigma_2\sigma_3\rangle. \tag{2.5}$$

The inequality (2.5) is true for any size system including an infinite system. If we now let $h\to 0$ and use the translational and rotational symmetries of the system, we have

$$\langle\sigma_0\rangle\leqslant 4T_2\langle\sigma_0\rangle+4T_4\langle\sigma_1\sigma_2\sigma_3\rangle. \tag{2.6}$$

The steps to this point are of the same basic type as in Monroe (1985). The new feature is the thermal average $\langle\sigma_1\sigma_2\sigma_3\rangle$ which does not appear if only pair interactions are present. We can reduce this three-site thermal average to a single site by using the inequality

$$\langle\sigma_1\rangle\geqslant\langle\sigma_1\sigma_2\sigma_3\rangle. \tag{2.7}$$

This inequality and other similar inequalities needed can be established by the duplicate variable method of proving correlation inequalities; see Ellis and Monroe (1975), Ellis *et al* (1976) and Sylvester (1976). Messager and Miracle-Sole (1977) have used this method to establish some of the inequalities needed here. We will defer to § 3 the proofs of all inequalities similar to (2.7) needed for our bounds on T_c. Using (2.7) along with the translational symmetry of the system we have

$$\langle\sigma_0\rangle\leqslant(4T_2+4T_4)\langle\sigma_0\rangle. \tag{2.8}$$

If the term in parentheses is less than one then $\langle\sigma_0\rangle=0$ for the infinite system with $h\to 0$ and hence we have no phase transition. We plot in figure 2 the bounds on T_c found from (2.8).

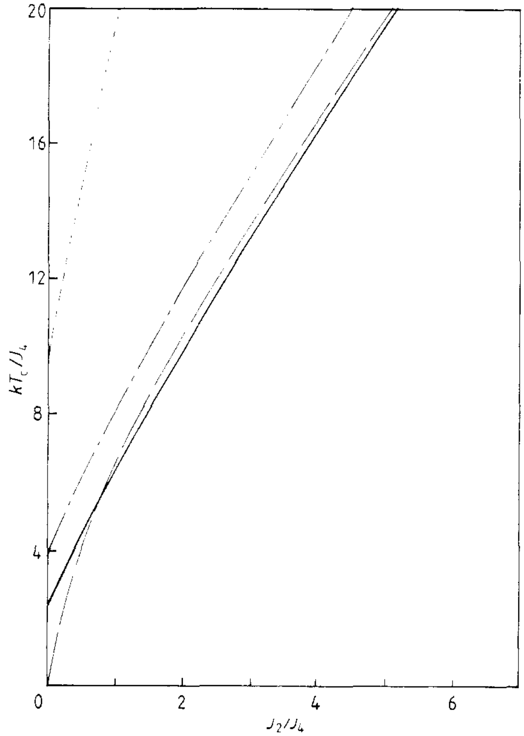

Figure 2. Upper bounds for KT_c/J_4 for the square lattice system. --- (— — —): result from Horiguchi and Morita 1985a (1985b). —·—: results from (2.8); ——: result from (2.11).

The bound given by (2.8) can be improved by going back to the identity (2.4), and changing our approach to (2.8) in two ways. First, setting each of the B in (2.4) equal to one is equivalent to setting each of the thermal averages in each expression for a particular B equal to zero. Clearly these thermal averages are greater than zero and a better lower bound can be established for them. This was done for the pair interaction case but will not be done here. Rather we will emphasise a second basic way of improving the inequality. It consists of continuing with the process of deleting interactions beyond the level arrived at in (2.4).

Specifically we take the thermal averages in the numerators of the last eight terms of (2.4) and begin deleting all interactions involving one of the sites in the thermal average, for example, starting with $\langle\sigma_1\rangle_{01}$ we delete the remaining three-pair interaction terms and the four four-body interactions involving spin σ_1. This gives, setting all B terms equal to one,

$$\langle\sigma_1\rangle_{01} \leq T_2\langle\sigma_2\rangle_{12,01} + T_2\langle\sigma_{10}\rangle_{110,12,01} + T_2\langle\sigma_8\rangle_{18,110,12,01} + T_4\langle\sigma_0\sigma_2\sigma_3\rangle'_\alpha$$
$$+ T_4\langle\sigma_2\sigma_9\sigma_{10}\rangle'_{\varepsilon\alpha} + T_4\langle\sigma_8\sigma_{10}\sigma_{11}\rangle'_{\rho\varepsilon\alpha} + T_4\langle\sigma_0\sigma_7\sigma_8\rangle'_{\delta\rho\varepsilon\alpha} + \tau \tag{2.9}$$

where the prime on the thermal average indicates that the four-pair interactions are deleted. With a thermal average involving three sites such as $\langle\sigma_1\sigma_2\sigma_3\rangle_\alpha$ we have

$$\langle\sigma_1\sigma_2\sigma_3\rangle'_\alpha \leq T_2\langle\sigma_3\rangle'_{12,\alpha} + T_2\langle\sigma_2\sigma_3\sigma_{10}\rangle'_{110,12,\alpha} + T_2\langle\sigma_8\sigma_2\sigma_3\rangle'_{18,110,12,\alpha} + T_4\langle\sigma_3\sigma_9\sigma_{10}\rangle'_{\varepsilon,\ldots,\alpha}$$
$$+ T_4\langle\sigma_2\sigma_3\sigma_8\sigma_{10}\sigma_{11}\rangle'_{\rho,\varepsilon,\ldots,\alpha} + T_4\langle\sigma_0\sigma_2\sigma_3\sigma_7\sigma_8\rangle'_{\delta,\rho,\varepsilon,\ldots,\alpha} + \tau\langle\sigma_2\sigma_3\rangle_{\delta,\ldots,\alpha} \tag{2.10}$$

where all sites and interactions are labelled as in figure 1. These expressions can now be used to bound the thermal averages in the last eight terms of (2.4).

Placing these terms into (2.4) results in an improved bound on T_c. One should notice that we now need an expanded set of correlation inequalities since we now have some thermal averages containing the product of five spin variables. We postpone, as with (2.7), the proof of these inequalities until § 3. We gain a further improvement if we take the thermal averages on the right-hand side of (2.9) and (2.10) and the other thermal averages generated from the six other terms of (2.4) and delete all interactions about one site in each of these terms. We then think of this as a third generation bound. There is in principle nothing to stop us from continuing in this manner although, as seen already in (2.9) and (2.10), to keep track of the deleted interactions becomes a very tedious task. For this reason we stop at the level of the third generation inequality which is

$$\langle\sigma_0\rangle \leqslant (36T_2^3 + 122T_2^2T_4 + 93T_2T_4^2 + 19T_4^3)\langle\sigma_0\rangle. \tag{2.11}$$

If as in (2.8) the term in parentheses is less than one then there is no spontaneous magnetisation.

The bound given by (2.11) is plotted in figure 2 along with the results from (2.8) and from Horiguchi and Morita (1985a, b). Comparison of the results depends on the ratio of J_2 to J_4. In Horiguchi and Morita (1985b), for J_2/J_4 close to zero their results are significantly better since they have as $J_2/J_4 \to 0$ that $T_c \to 0$ which we are not able to establish. However for intermediate values, for example $J_2/J_4 = 1$ and $J_2/J_4 = 5$, they obtain $kT_c/J_4 \leqslant 6.73$ and $kT_c/J_4 \leqslant 20.03$, respectively, while from (2.11) one has $kT_c/J_4 \leqslant 6.41$ and $kT_c/J_4 \leqslant 19.67$, respectively. Eventually for large enough values of J_2/J_4 the results of Horiguchi and Morita (1985b) will again become better. This is because when $J_2/J_4 \to \infty$ we have only the pair interaction system and Horiguchi and Morita use the exact results known for this system. This particular aspect of their method will be discussed in § 4.

We now consider as a second example of our method a triangular lattice system with three-body interactions J_3 on each elementary triangle of the lattice and an external magnetic field. To have the symmetry necessary for the proof of the correlation inequalities we take the set of sites making up the system to be hexagonal in shape as in figure 3. We take the system to have free boundary conditions. We let this set of

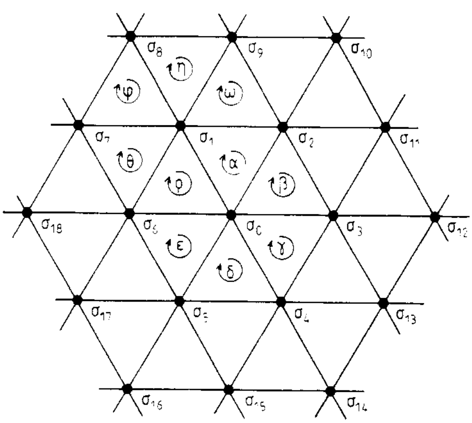

Figure 3. The triangle lattice with sites and three-body interactions labelled.

sites become infinite and then let the external magnetic field go to zero. We label the sites as shown in figure 3 and as before use Greek letters to represent specific three-body interactions.

Deleting interactions around the zeroth site we have

$$\langle\sigma_0\rangle=\frac{\tau}{B_\rho\ldots B_\alpha}+\frac{T_3\langle\sigma\sigma_1\rangle_{\rho\ldots\alpha}}{B_\rho\ldots B_\alpha}+\frac{T_3\langle\sigma_5\sigma_6\rangle_{\varepsilon\ldots\alpha}}{B_\varepsilon\ldots B_\alpha}+\frac{T_3\langle\sigma_4\sigma_5\rangle_{\delta\ldots\alpha}}{B_\delta\ldots B_\alpha}$$
$$+\frac{T_3\langle\sigma_3\sigma_4\rangle_{\gamma\beta\alpha}}{B_\gamma B_\beta B_\alpha}+\frac{T_3\langle\sigma_2\sigma_3\rangle_{\beta\alpha}}{B_\beta B_\alpha}+\frac{T_3\langle\sigma_1\sigma_2\rangle_\alpha}{B_\alpha} \tag{2.12}$$

where $B_\rho=1+T_3\langle\sigma_0\sigma_1\sigma_6\rangle_{\rho\ldots\alpha}$, $B_\varepsilon=1+T_3\langle\sigma_0\sigma_5\sigma_6\rangle_{\varepsilon\ldots\alpha}$, etc, and $T_3=\tanh(\beta J_3)$. The symbol τ is defined as before. Again to obtain a bound we set each $B=1$. Now a new feature has arisen in that in the numerators of the last six terms of (2.12) we have thermal averages of a product of an even number of spins. Such thermal averages cannot be bounded by the methods of § 3. However, if we take the thermal averages and delete interactions involving one of the pair of sites in each average we generate a second generation bound. The thermal averages in this set of deletions all contain an odd number of spins: for example, starting with the term $\langle\sigma_1\sigma_2\rangle_\alpha$ we have

$$\langle\sigma_1\sigma_2\rangle_\alpha\leqslant\tau\langle\sigma_2\rangle_{\omega\ldots\alpha}+T_3\langle\sigma_9\rangle_{\omega\ldots\alpha}+T_3\langle\sigma_2\sigma_8\sigma_9\rangle_{\eta\ldots\alpha}+T_3\langle\sigma_2\sigma_7\sigma_8\rangle_{\phi\ldots\alpha}$$
$$+T_3\langle\sigma_2\sigma_6\sigma_7\rangle_{\theta\rho\alpha}+T_3\langle\sigma_2\sigma_6\sigma_0\rangle_{\rho\alpha} \tag{2.13}$$

where as before the B have been set equal to one. The thermal averages involving three spins can be bounded from above by single-spin thermal averages as was done for the square lattice systems. However, because periodic boundary conditions are not used each single-spin thermal average is not equal to every other single-spin thermal average. However, as a direct consequence of the set of inequalities in § 3 we show

$$\langle\sigma_0\rangle\geqslant\langle\sigma_i\rangle \tag{2.14}$$

for some subset of sites i which includes all the sites of our one-site thermal averages. We then have

$$\langle\sigma_0\rangle\leqslant 25\,T_3^2\langle\sigma_0\rangle \tag{2.15}$$

and we thus have a bound of $KT_c/J_3\leqslant 4.93$. Baxter and Wu's exact solution gives $KT_c/J_3=2.27$. The bound can be improved by going to a third generation inequality or higher. However, since this model has been solved exactly we have not done so. Our intent in considering the above model was to show how the method can be applied to a spin system with odd-body ferromagnetic interactions in contrast to the Horiguchi and Morita method.

3. Correlation inequalities

The correlation inequalities we used in § 2 are related to a set of inequalities established by Messager and Miracle-Sole (1977). Their proof as well as ours is based on the duplicate variable method (see Ellis and Monroe 1975, Sylvester 1976). Since the symmetry properties of the individual systems are crucial we must prove the necessary inequalities for each system separately. We begin with the square lattice.

For simplicity we consider a four site by four site system. The method works for any $2n$ by $2n$ site system. We divide the system into two halves either horizontally or vertically and label the sites as shown in figure 4. Note that for every site i there is a site $\bar{i}$. For any set D of lattice sites all on the unbarred half of the system we write

$$\langle \sigma_D \rangle = \left\langle \prod_{i \in D} \sigma_i \right\rangle. \tag{3.1}$$

Then $\sigma_{\bar{D}}$ is the product of the spin variables found by reflecting D about the vertical broken line in figure 4.

Theorem 1. For a system with Hamiltonian (2.1) on a square lattice with periodic boundary conditions

$$\langle \sigma_A \sigma_B \rangle \geqslant \langle \sigma_A \sigma_{\bar{B}} \rangle. \tag{3.2}$$

Proof. Rewriting (3.2) we need to establish that

$$\langle \sigma_A (\sigma_B - \sigma_{\bar{B}}) \rangle \geqslant 0. \tag{3.3}$$

Define for each site i

$$t_i = \tfrac{1}{2}(\sigma_i + \sigma_{\bar{i}}) \qquad q_i = \tfrac{1}{2}(\sigma_i - \sigma_{\bar{i}}). \tag{3.4}$$

We can re-express the Hamiltonian in terms of the t and q and in doing so the Hamiltonian becomes a polynomial in the t and q with negative coefficients and some constant terms. In the evaluation of the thermal average we have the factor $\exp(-\beta H)$. The exponent of this factor is a polynomial involving terms of t and q with positive coefficients and constant terms. This is also true of the term $\sigma_A(\sigma_B - \sigma_{\bar{B}})$ when written in terms of the t and q. We expand all the exponential terms involving the t and q and factor the expression collecting terms by site. Now the problem is reduced to showing

$$\sum_{\sigma_i, \sigma_{\bar{i}}} t_i^m q_i^n \geqslant 0. \tag{3.5}$$

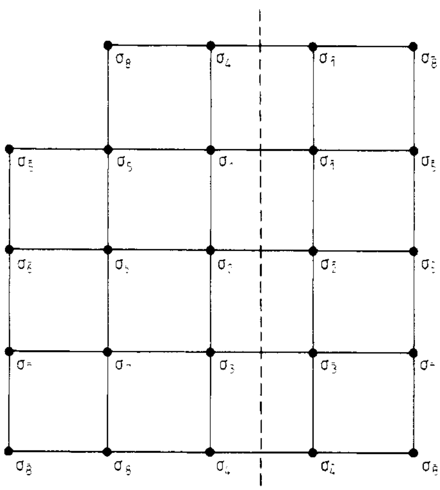

Figure 4. A square lattice system with sites labelled for the correlation inequality proof. Some sites are shown twice so that one can more easily see the interactions due to the periodic boundary conditions.

This summation vanishes by symmetry unless both m and n are even in which case the summation is clearly positive. Hence we have the inequality (3.2) since it is just made up of sums of products of the form (3.5), all of which are non-negative.

Now we consider the specific inequalities used in § 2 such as (2.7). For (2.7) split the system into barred and unbarred halves by a horizontal line between σ_2 and σ_3 in figure 1. Then we have $\sigma_1 = \sigma_1$, $\sigma_2 = \sigma_2$ and $\sigma_3 = \sigma_{\bar{2}}$ and therefore by the theorem and $\sigma_i^2 = 1$ we have

$$\langle \sigma_1 \rangle = \langle \sigma_1 \sigma_2 \sigma_2 \rangle \geqslant \langle \sigma_1 \sigma_2 \sigma_{\bar{2}} \rangle = \langle \sigma_1 \sigma_2 \sigma_3 \rangle. \tag{3.6}$$

When we have an expression involving five spins such as $\langle \sigma_2 \sigma_3 \sigma_8 \sigma_{10} \sigma_{11} \rangle$ in (2.10) then we may have to use the process twice. First again divide the system between sites 2 and 3 in figure 1, then

$$\langle \sigma_8 \sigma_{10} \sigma_{11} \rangle = \langle \sigma_2 \sigma_2 \sigma_8 \sigma_{10} \sigma_{11} \rangle \geqslant \langle \sigma_2 \sigma_{\bar{2}} \sigma_8 \sigma_{10} \sigma_{11} \rangle = \langle \sigma_2 \sigma_3 \sigma_8 \sigma_{10} \sigma_{11} \rangle. \tag{3.7}$$

Now divide the system vertically into two halves along a line between σ_{10} and σ_{11} so that $\sigma_{\overline{11}} = \sigma_{10}$ then

$$\langle \sigma_8 \rangle = \langle \sigma_8 \sigma_{11} \sigma_{11} \rangle \geqslant \langle \sigma_8 \sigma_{11} \sigma_{\overline{11}} \rangle = \langle \sigma_8 \sigma_{11} \sigma_{10} \rangle. \tag{3.8}$$

All odd-body inequalities contained in the establishment of the bounds on T_c for the square lattice can be reduced to a single-spin thermal average. Because of the periodic boundary conditions each single-spin thermal average is equivalent to any other so we can reduce everything to $\langle \sigma_0 \rangle$.

For the triangular system we can use the same approach except for some minor modifications due to the different symmetry and the different boundary conditions used for this case. We therefore only mention the necessary modifications. For simplicity we consider the 19-site system in figure 5. We divide the system into two halves along *any* row of sites. Label the sites on the larger half of the system as unbarred sites and their reflection is then the barred sites. An example of this is shown in figure 5.

Defining t and q as in equation (3.4) we have, when re-expressing the Hamiltonian and $\sigma_A(\sigma_B - \sigma_{\bar{B}})$, that all terms involving products of t and q have non-negative coefficients. Collecting terms site by site and having (3.5) we have the inequality (3.2)

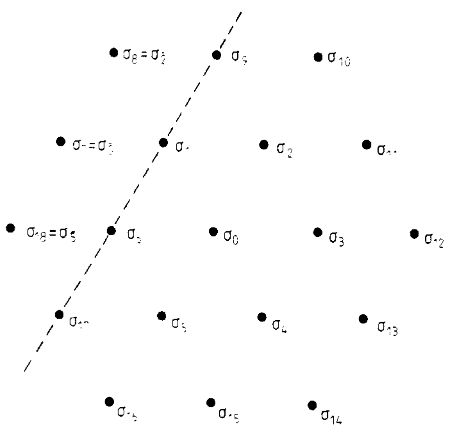

Figure 5. A triangular lattice system with sites labelled for the correlation inequality proof.

for the triangular lattice system. This inequality we use in two different ways in obtaining the bounds on T_c such as that following (2.15). First we reduce any three-site thermal average to a single-site thermal average in the same manner as was done for the square lattice. However, because of the change in boundary conditions we do not have the equivalency of each single-site thermal average. We can, however, using inequality (3.2), show that $\langle\sigma_0\rangle \geqslant \langle\sigma_i\rangle$ for i such that it is a reflection of σ_0 about some allowed division of our system into halves. An example of such a site is $i=7$. Then by the labelling of the sites in figure 5 we have

$$\langle\sigma_0\rangle = \langle\sigma_0\sigma_0\sigma_0\rangle \geqslant \langle\sigma_0\sigma_0\sigma_{\bar{0}}\rangle = \langle\sigma_7\rangle. \tag{3.9}$$

When reducing our three-site thermal averages in the second generation bound of (2.15) only the sites $i=7$, 9, 11, 13, 15 and 17 appear and these can all be bounded by $\langle\sigma_0\rangle$ using the above method.

4. Conclusion

We have tried to show the flexibility of a method, previously used to find upper bounds on T_c for ferromagnetic pair interaction systems, when being applied to systems with ferromagnetic many-body interactions. Our emphasis has been more on the method than pushing for the best numerical results. We have therefore restricted our calculations to ones involving rather a small number of terms. With this number of terms our bounds are comparable to the best bounds known to us for these systems.

It is clear from the earlier sections that one of the characteristics of the method is that a sequence of improving bounds can be achieved. This can be done in two ways. First one can go to higher generations of deleted bonds which means then considering more and more terms. Second, it can be done by not using the very crudest lower bound of $B \geqslant 1$ for all the various B generated, which is equivalent to bounding all the thermal averages in the B by zero. Rather we can easily obtain non-zero lower bounds for these thermal averages by calculating them in small systems as was done for the case of the pair interaction systems. We know from the GKS inequalities the bigger these small systems the better the bound.

Finally, since throughout the paper we have, when possible, compared our results with the Horiguchi and Morita results we bring out one final point for comparison. Their method bounds the many-body system by a system with only pair interactions. Then use is made of the available results for this pair interaction system. The better the results for the pair interaction system the better the results for the many-body interaction system. In the case of the two-dimensional square lattice one has the exact solution for T_c but for other lattice systems, especially three-dimensional systems, similar results will not in general be available. The method presented in this paper is self-contained in the sense that no results on a critical temperature established elsewhere are used here.

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